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Damage Localization from the Null Space of Changes in the Transfer Matrix

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A new theorem connecting changes in transfer matrices to the spatial location of stiffness related damage is presented. The theorem states that the span of the null space of the change in the transfer matrix (ΔG) contains vectors that are Laplace transforms of excitations for which the dynamic stress field is identically zero over the portion of the domain where the damage is located. A corollary states that if ΔG proves rank deficient at any s it is guaranteed to be rank deficient throughout the plane. It is shown that a sufficient condition for rank deficiency is that the number of independent measurements be larger than the rank of the change in the stiffness matrix resulting from damage. Implementation of the theorem in the time domain involves evaluation of the null space of ΔG along a discretized Bromwich contour, numerical inversion of the resulting s-functions and evaluation of the dynamic response to the computed signals to identify the regions where the stress field is zero. The theorem is most efficiently implemented, however, in the s-domain, where numerical Laplace inversion is avoided and the number of evaluations of the null of ΔG is sharply reduced. Guidelines for implementation of the theorem in the presence of inevitable errors in ΔG and in the model used to compute the stress fields are presented.

Nomenclature

C_D	=	damping matrix
G	=	experimental transfer matrix in the reference state $\in C^{q\times q}$
G_m	=	transfer matrix of the model of the system $\in C^{n\times n}$
$\tilde{G}$	=	experimental transfer matrix in the potentially damaged state $\in C^{q\times q}$
K	=	stiffness matrix $\in R^{n\times n}$
ℓ	=	vector in the span of the null space of $\Delta G \in C^{q\times 1}$
M	=	mass matrix $\in R^{n\times n}$
m	=	number of output measurements
n	=	number of degrees of freedom in the nominal model
p	=	rank of ΔK
q	=	union of all the input and output coordinates
r	=	number of independent inputs
s	=	Laplace variable
y	=	displacement vector at all coordinates
ΔG	=	change in the transfer matrix due to damage $\in C^{q\times q}$
ΔK	=	change in the stiffness matrix due to damage $\in R^{n\times n}$

I. Introduction

A widely used framework in vibration based identification of damage is the so called before and after strategy. In this scheme, damage (if any) is ascertained by contrasting data-validated system characterizations obtained at the start and the end of a given time interval. Methods that operate within this framework can be classified as belonging to one of two large groups: 1) those where the system characterization is obtained entirely from the measurements [1–3] and 2) techniques in which a finite element model is used to represent the structure [4,5]. Methods that belong to the first group can be effective when the objective is restricted to discriminating between damage or no damage but are much less attractive if localization or quantification is needed, because, in the absence of a model, mapping of changes in identification to the physical domain must be done heuristically, and at a scale that is essentially determined by the sensor grid. Because this paper focuses on the localization of damage, data-driven techniques are not considered further.

The fundamental difficulty in a model-based characterization of damage lies in the fact that fitting the model to the data proves ill-conditioned because the free parameter space needed to consider all the possible damage scenarios is typically large. The key to resolve this difficulty resides in decoupling localization from quantification and finding ways to solve the localization component without the need to select a free parameter space. The foregoing is particularly so because damage is local in nature and, once located, can be parameterized with a small number of parameters, rendering the subsequent quantification stage well conditioned.

Methods for decoupling localization information from severity have traditionally used orthogonality between a subspace that can be computed from the model and a feature derived from the measurements. Choudhury and He [6], for example, use a subspace given by the inverse of the frequency response matrix (FRM) and the experimental feature is the difference between frequency response functions (FRFs) from the reference and the damaged states. Assuming that damage is restricted to changes in stiffness, the product of these two entities can be shown to be zero at all degrees of freedom that are not affected by the damage. Another method, known as the best achievable eigenvector (BAE) [7–9] technique, is a forward method that interrogates one element at a time and announces damage when the span of a subspace that depends on the element being considered contains the identified eigenvectors. Although extension to the case of multiple damaged elements has been discussed in the literature, the theoretical construct of the BAE technique is restricted to single damage. Note that the BAE approach can be cast in the orthogonality format described at the top of the paragraph by replacing the element dependent model subspace by its null.

An important limitation shared by both of the techniques mentioned is the fact that the experimental feature (the FRFs in [6] or the mode shapes in [7–9]) needs to be available at all the coordinates of the analytical model. Because this requirement is never satisfied, the missing entries in the experimental features have to be estimated using coordinate expansion techniques [10,11] and approximation is introduced. A localization method that operates without regard to differences between the coordinates in the model and the measured

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degrees of freedom, however, is the damage locating vector (DLV) approach [12]. The DLV method locates the collection of potentially damaged elements as those having zero stress when the structure is statically loaded with vectors from the null space of the change in experimentally extracted flexibility matrices.

The contribution presented in this paper has the DLV technique as a precursor and consists on a new theorem that connects the null space of the change in the transfer matrix (ΔG) to the location of stiffness related damage. The theorem, designated as the dynamic damage locating vector (DDLV) theorem states that the null space of ΔG contains vectors that are Laplace transforms of dynamic excitations that induce zero stress over the damaged region. The DDLV theorem can be viewed as extending the null space approach in the DLV method from statics to dynamics. Implementation of the DDLV theorem in the time domain involves 1) processing of the data to derive transfer matrices, 2) selection of a Bromwich contour and evaluation of the null space of ΔG at each point of the discretization, 3) numerical Laplace inversion of the resulting s-functions, and finally 4) computation of the stress field history for the signals in step 3, to identify the regions of zero stress. As is evident from this outline, a time domain implementation of the DDLV theorem is computationally expensive and potentially vulnerable to difficulties arising from the numerical Laplace inversion [13]. By focusing on the Laplace transform of the stress field instead of its time history, however, the need for numerical inversion is eliminated and computational efficiency is realized.

It is worth noting that the reason why the DDLV theorem is presented in terms of transfer matrices and not FRMs is because robustness in the presence of errors in the model and in the identification is not, in general, optimal on the imaginary line. Whereas a numerical study of robustness is outside the scope of this paper, the merit of keeping the flexibility offered by the Laplace variable (instead of taking $s = i\omega$) is discussed in the body. The rest of the paper is organized as follows: Sec. II presents the DDLV theorem and its proof; Sec. III reviews the derivation of transfer matrices from measured data with attention given to the conversion from force-acceleration to force-displacement and illustrates a form that allows consideration of structures with rigid body modes. Sections IV and V contain the details of the time domain and s-domain implementations, and Sec. VI concludes the paper with a critical review of the material presented.

II. DDLV Theorem and Its Proof

The experimental transfer matrix G is a map from r inputs to m outputs in the s-domain. Provided that $m \cap r \neq \varnothing$ it is possible, by taking advantage of reciprocity, to expand G to a map from $\boldsymbol{q} \to \boldsymbol{q}$ where $q = m \cup r$ [14,15]. In the derivation that follows we assume that G refers to the $\boldsymbol{q} \to \boldsymbol{q}$ map and that the inputs are forces and the outputs displacements. The connection between the matrices of a state-space realization obtained from displacement velocity or acceleration measurements and the force-displacement transfer matrix used in the derivation is clarified in Sec. III.

A. DDLV Theorem

Consider a finite dimensional representation of a linear viscously damped structure in two states: one where the stiffness matrix is K and the other $(K + \Delta K)$, where ΔK derives exclusively from reductions in some of the parameters that determine the stiffness. Designate the collection of elements that contribute to ΔK as constituting the physical domain Ω_D and assume that $f_{(t)}$ is a time history of loads acting at a set of coordinates q with $\sigma_{D,t}$ being the associated stress field over Ω_D. The DDLV theorem states that if $f_{(t)}$ is such that $\sigma_{D,t} \equiv 0$ then $\mathcal{L}(f_{(t)}) = \mathrm{Null}(\Delta G) \cdot \nu$, where ΔG is the change in the transfer matrix for the q coordinates and ν is arbitrary; in other words, the vectors that span the null space of the change in the transfer matrix are Laplace transforms of dynamic loads for which the associated stress field is zero over the damaged region.

A somewhat less direct but more precise statement makes clear that the theorem does not forbid zero stresses over elements that are not damaged, namely: the vectors that span the null space of the change in the transfer matrix are Laplace transforms of loads for which the stress field is nonzero at locations where there is no damage.

B. Proof of the DDLV Theorem

The progression is as follows: 1) identify expressions for dynamic loads that induce identical response in the damage and the undamaged states, 2) show that the loads of step 1 are associated with stress fields that are zero over the damaged region, 3) examine the conditions for which there is a subset of the loads from step 1 that have entries that are zero at all unmeasured coordinates, and finally 4) show that if the loads in step 3 exist, their Laplace transforms are in the span of the null space of ΔG. We conclude these preliminaries by noting that although the DDLV theorem does not say anything regarding when a null space is anticipated, clarifying the conditions for which ΔG is rank deficient is important and is done within the proof. There are three transfer matrices involved in the derivation, namely, the experimental transfer matrices of the undamaged and damaged states, G and $\tilde{G}$, of dimension q, and the transfer matrix associated with the model in the undamaged state G_m, of dimension n; needless to say, in most applications $n \gg q$.

Given the assumption of finite dimensionality and linearity one can write

$$M\ddot{y}_{(t)} + C_D\dot{y}_{(t)} + Ky_{(t)} = f_{(t)} \tag{1}$$

and

$$M\ddot{\tilde{y}}_{(t)} + C_D\dot{\tilde{y}}_{(t)} + (K + \Delta K)\tilde{y}_{(t)} = \tilde{f}_{(t)} \tag{2}$$

where the tilde is used to distinguish the damaged condition. Subtracting Eq. (2) from Eq. (1) one gets

$$\Delta K\hat{y}_{(t)} = 0 \tag{3}$$

which shows that $\hat{y}_{(t)}$ exists if the change in the stiffness matrix due to damage is rank deficient. From Eq. (3) one can write

$$\hat{y}_{(t)} = Qg_{(t)} \tag{4}$$

where $Q \in R^{n\times(n-p)}$ is a basis for the null space of ΔK, $g_{(t)} \in R^{(n-p)\times 1}$ is an arbitrary time dependent vector, and p is the rank of ΔK. From the result in Eq. (4) one finds that all the loadings that produce identical response are given by

$$\hat{f}_{(t)} = MQ\ddot{g}_{(t)} + C_DQ\dot{g}_{(t)} + KQg_{(t)} \tag{5}$$

For any load in Eq. (5) the strain energy in the undamaged and damaged states are

$$U_{(t)} = \frac{1}{2}\hat{y}_{(t)}^T K\hat{y}_{(t)} \tag{6}$$

$$\tilde{U}_{(t)} = \frac{1}{2}\hat{y}_{(t)}^T (K + \Delta K)\hat{y}_{(t)} \tag{7}$$

and from the result in Eq. (3) [and the expressions in Eqs. (6) and (7)] one concludes that for the loads in Eq. (5)

$$U_{(t)} = \tilde{U}_{(t)} \tag{8}$$

To show that the result in Eq. (8) implies zero stress over the damaged region consider the strain energy for the undamaged and damaged states in terms of element contributions, namely,

$$U_{(t)} = \frac{1}{2}\left[\sum_{\Omega_U}\hat{e}_j^T k_j\hat{e}_j + \sum_{\Omega_D}\hat{e}_i^T k_i\hat{e}_i\right] \tag{9}$$

$$\tilde{U}_{(t)} = \frac{1}{2}\left[\sum_{\Omega_U} \hat{e}_j^T k_j \hat{e}_j + \sum_{\Omega_D} \hat{e}_i^T \tilde{k}_i \hat{e}_i\right] \tag{10}$$

where $\hat{e}_j$ is the portion of the vector $\hat{y}_{(t)}$ that corresponds to nodal displacements for element j and the matrices k and $\tilde{k}$ are the undamaged and damaged element stiffness, respectively. Substituting Eqs. (9) and (10) into Eq. (8) gives

$$\sum_{\Omega_D} \hat{e}_j^T k_j \hat{e}_j = \sum_{\Omega_D} \hat{e}_j^T \tilde{k}_j \hat{e}_j \tag{11}$$

Finally, expressing the damaged element stiffness matrix as the sum of the undamaged matrix plus a change, namely,

$$\tilde{k}_j = k_j + \delta k_j \tag{12}$$

one finds that

$$\sum_{\Omega_D} \hat{e}_j^T \delta k_j \hat{e}_j = 0 \tag{13}$$

which can only be satisfied if each term is zero because the expression is a quadratic form and δk_j is, by definition, negative semidefinite. One concludes that for the loads defined by Eq. (5) the strain energy over Ω_D, and as a consequence the stress field, are identically zero.

It is not difficult to see that one cannot compute an arbitrary member of the loads in Eq. (5) from the data because these loads are defined over the full set of coordinates. What can be extracted, however, is the subset (if it exists) that has zero entries at unmeasured coordinates. For convenience in the subsequent discussion assume that the DOF are ordered such that the experimental ones, y_q, appear first, namely, $y = [y_q y_u]^T$, where y_u are the "unmeasured" coordinates. Taking a Laplace transform of Eq. (5) and restricting the results to functions with $g_{(0)} = \dot{g}_{(0)} = 0$ one gets

$$P_s g_{(s)} = \hat{f}_{(s)} \tag{14}$$

where

$$P_{(s)} = (Ms^2 + C_D s + K)Q \tag{15}$$

with $P_{(s)} \in C^{n\times(n-p)}$. Imposing the requirement that the loads in Eq. (14) be identically zero over the unmeasured set gives

$$\begin{bmatrix} P_{q,(s)} \\ P_{u,(s)} \end{bmatrix} \{g_{(s)}\} = \begin{bmatrix} \ell_{(s)} \\ 0 \end{bmatrix} \tag{16}$$

where $P_q \in C^{q\times(n-p)}$ and $P_u \in C^{(n-q)\times(n-p)}$. Because $g_{(s)}$ is arbitrary (except for the behavior required as $s \to \infty$ determined by the assumed zero initial conditions) one concludes that Eq. (16) is satisfied by

$$g_{(s)} = \mathrm{Null}(P_{u,(s)}) \cdot h \tag{17}$$

and the vectors $\ell_{(s)}$ are

$$\ell_{(s)} = \Gamma_{(s)} \cdot h \tag{18}$$

where h is an arbitrary vector of appropriate dimension and

$$\Gamma_{(s)} = P_{q,(s)} \cdot \mathrm{Null}(P_{u,(s)}) \tag{19}$$

Noting $P_{u,(s)} \in C^{(n-q)\times(n-p)}$, one has Null $(P_u) \in C^{(n-p)\times z}$; where $z \geq q - p$ and, given that $\Gamma \in C^{q\times z}$, it follows that rank $(\Gamma) \leq z$. Whereas the mathematics only guarantee rank $(\Gamma) \leq z$, there does not appear to be any reason why Γ should not be full column rank, so one anticipates that the dimension of the basis that contains the ℓ vectors equals z. In summary: vectors ℓ exist provided q is larger than the rank of ΔK.

C. Useful Lemma

Lemma: If damage is restricted to changes in stiffness, the rank of the change in the transfer matrix (defined over any set of coordinates) at any point in the s-plane is no larger than the rank of ΔK.

Proof: Because the objective is to show that the rank of ΔG does not exceed p we consider the extreme case where the transfer matrix is defined over the n coordinates of the nominal model. For this condition one can write

$$G_{(s)}^{-1} = X_{(s)} + K \tag{20}$$

and

$$\tilde{G}_{(s)}^{-1} = X_{(s)} + K + \Delta K \tag{21}$$

where $X_{(s)} = Ms^2 + C_D s$. Subtracting Eq. (20) from (21) gives

$$\tilde{G}_{(s)}^{-1} - G_{(s)}^{-1} = \Delta K \tag{22}$$

therefore,

$$\tilde{G}_{(s)} = \left(G_{(s)}^{-1} + \Delta K\right)^{-1} \tag{23}$$

Using the Sherman–Morrison–Woodbury formula [16], Eq. (23) can be written as

$$\tilde{G}_{(s)} = G_{(s)} - G_{(s)}\Delta K\, (I + G_{(s)}\Delta K)^{-1} G_{(s)} \tag{24}$$

from where one has

$$\Delta G_{(s)} = -G_{(s)} \Delta K N G_{(s)} \tag{25}$$

with

$$N = (I + G_{(s)}\Delta K)^{-1} \tag{26}$$

and because the rank of a product of matrices is not larger than the smallest rank of any of the individual factors it follows from Eq. (25) that the rank of $\Delta G_{(s)}$ is not larger than p, completing the proof.

D. Computation of $\ell_{(s)}$ from the Null of the Change in the Transfer Matrix

To show that $\ell_{(s)}$ can be computed from the experimental ΔG assume again that the coordinates are ordered $y = [y_q y_u]^T$. Given the definition of $\ell_{(s)}$ one can write

$$\begin{bmatrix} G_{q,q} & G_{q,u} \\ G_{q,u}^T & G_{u,u} \end{bmatrix} \begin{bmatrix} \ell_{(s)} \\ 0 \end{bmatrix} = \begin{Bmatrix} \hat{y}_q \\ \hat{y}_u \end{Bmatrix} \tag{27}$$

and

$$\begin{bmatrix} \tilde{G}_{q,q} & \tilde{G}_{q,u} \\ \tilde{G}_{q,u}^T & \tilde{G}_{u,u} \end{bmatrix} \begin{bmatrix} \ell_{(s)} \\ 0 \end{bmatrix} = \begin{Bmatrix} \hat{y}_q \\ \hat{y}_u \end{Bmatrix} \tag{28}$$

therefore

$$\begin{bmatrix} \Delta G_{q,q} & \Delta G_{q,u} \\ \Delta G_{q,u}^T & \Delta G_{u,u} \end{bmatrix} \begin{bmatrix} \ell_{(s)} \\ 0 \end{bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{29}$$

which shows that

$$\ell_{(s)} = \mathrm{Null}\begin{bmatrix} \Delta G_{q,q} \\ \Delta G_{q,u}^T \end{bmatrix} \cdot \vartheta \tag{30}$$

where ϑ is arbitrary. What is stated in the DDLV theorem, however, is that

$$\ell_{(s)} = \mathrm{Null}(\Delta G_{q,q}) \cdot \nu \tag{31}$$

where ν is arbitrary. To show that Eq. (31) is valid we return to Eq. (25) and write it in partitioned form as

$$\Delta G_{(s)} = -\begin{bmatrix} G_q \\ G_u \end{bmatrix} \Delta KN \begin{bmatrix} G_q^T & G_u^T \end{bmatrix} \tag{32}$$

which, carrying out the multiplications, gives

$$\Delta G_{(s)} = -\begin{bmatrix} G_q(\Delta KNG_q^T) & \bullet \\ G_u(\Delta KNG_q^T) & \bullet \end{bmatrix} \tag{33}$$

where the terms that are not needed in the discussion have been replaced by dots. From inspection of Eq. (33) and (29) one concludes that the terms shown explicitly in Eq. (33) are the partitions in Eq. (30). The validity of Eq. (31) follows from the fact that the matrix in parenthesis in Eq. (33) $\in C^{n\times q}$ contains all the $\ell_{(s)}$ vectors (because its rank is p and its minimum dimension is q) and is a common post-multiplier to both entries in Eq. (30).

III. Transfer Matrix

In preceding sections we have operated on the premise that the transfer matrix relates forces to displacements. In practice, however, the most common output transducer is an accelerometer so it is convenient to have the force-displacement transfer matrix expressed in terms of the results that are obtained when accelerations are measured.

Assume that small amplitude vibration data have been used to obtain the matrices of a state-space realization. In continuous time one can write [17]

$$\dot{x} = A_c x + B_c u \tag{34}$$

where $A_c \in R^{2n\times 2n}$ is the system matrix and $B_c \in R^{2n\times r}$ is the input to state influence matrix. For displacement, velocity and acceleration measurements one has

$$y = C_d x \tag{35a}$$

$$\dot{y} = C_v x \tag{35b}$$

$$\ddot{y} = C_a x + Du \tag{35c}$$

where C_d, C_v, and $C_a \in R^{m\times 2n}$ are the state to output map for displacement velocity or acceleration measurements, $D \in R^{m\times r}$ is the direct transmission matrix, and x is the state vector. Taking a derivative of Eq. (35a), combining the result with Eq. (34) and contrasting the answer with Eq. (35b) one finds that

$$C_d A_c = C_v \tag{36}$$

Likewise, differentiating Eq. (35b), and performing some obvious substitutions one concludes that

$$C_d A_c^2 = C_a \tag{37}$$

Taking a Laplace transform of Eq. (34) and (35) and combining the results one gets that the force-displacement transfer matrix is

$$G = C_d[I\cdot s - A_c]^{-1} B_c \tag{38}$$

Finally, combining Eqs. (37) and (38) one gets

$$G = C_a A_c^{-2}[I\cdot s - A_c]^{-1} B_c \tag{39}$$

which gives the force to displacement transfer matrix when accelerations are measured. Equations (38) and (39), as well as the expression that holds when measurements are velocities can be compactly expressed as

$$G = C([I\cdot s - A_c]A_c^b)^{-1} B_c \tag{40}$$

where C is either C_a, C_v, or C_d, and $b = 0$, 1, or 2 for displacement, velocity, or acceleration sensing. As noted previously, the matrix in Eq. (40) is $m\times r$ but can be expanded, by taking advantage of reciprocity, to the union of all the experimental coordinates provided there is at least one collocated input output measurement [14,15].

IV. DDLV Theorem in the Time Domain

Whereas the DDLV theorem is most conveniently implemented in the s-domain, a discussion of the time domain implementation is instructive. Assume input and output signals have been recorded at two different times and used to compute transfer matrices. From these matrices, by simple subtraction, one obtains ΔG. Because we are considering here the ideal situation where ΔG is precise, there are two possibilities: either ΔG is full rank, in which case the number of sensors is not sufficient to allow localization of the damage using the DDLV theorem (at least with perfect zero stresses over the damaged region), or it is rank deficient, and the theorem holds strictly. Assuming ΔG is rank deficient, the dynamic loads that induce zero stress over the part of the domain that is damaged are given by

$$\ell_{(t)} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathrm{Null}(\Delta G)\cdot \nu \cdot e^{st}\,\mathrm{d}s \tag{41}$$

where ν is a vector of appropriate dimension and c, in the limits of the Bromwich contour, is taken to the right of all singularities. It is worth noting that if ν is selected arbitrarily, causality is not generally satisfied and the dynamic response (if computed from $t = 0$ starting at rest) will miss the contribution of some undetermined initial condition. Although causality can be enforced, it is much simpler to select ν arbitrarily and just recognize that there is a transient that must be allowed to die out before one decides on the regions that have zero stress. As is evident from Eq. (41), there are infinite realizations for the dynamic loads that locate the damaged region.

To illustrate application of the DDLV theorem in the time domain, consider a 3-DOF system with the arrangement and properties depicted in Fig. 1. Damping is taken as stiffness proportional with 5% of critical in the first mode and damage is simulated as loss or 50% in the stiffness of the spring connecting DOFs one and two. The system is assumed monitored in such a way that the transfer matrix for coordinates $\{1, 3\}$ is available. In this example the null space of ΔG is of dimension one, so ν is a scalar. Figure 2 shows a realization for $\ell_{(t)}$ obtained by taking $c = 0$ and $\nu = 1$ in Eq. (41), and the null vector as the standard output from MATLAB. The time histories of the forces in the springs are depicted in Fig. 3. As one anticipates from the theory, spring no. 2 (the damaged one) displays a zero force after the disruption due to the nonzero initial condition vanishes.

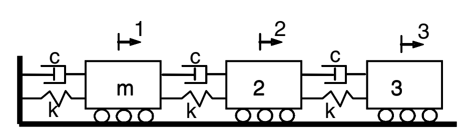

Fig. 1 Three-DOF system (in any set of consistent units: $m = 1$, $k = 100$, $c = 5.7$).

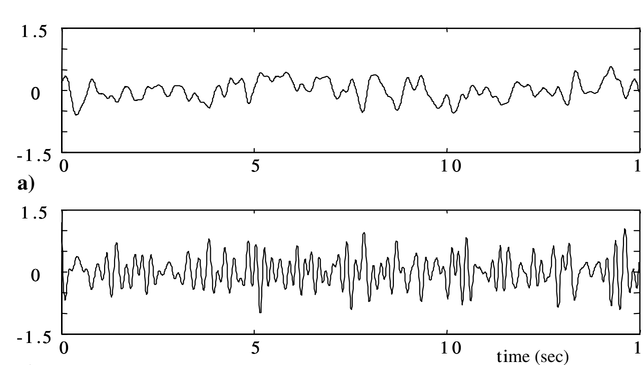

Fig. 2 Time history of damage locating loads computed from Eq. (41).

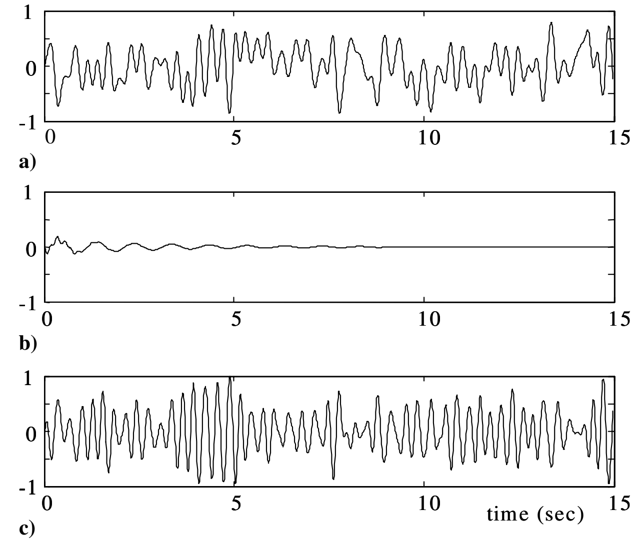

Fig. 3 Time history of spring forces: a) left, b) center, and c) right springs.

V. DDLV Theorem in the Laplace Domain

Although implementation of the DDLV theorem in the time domain is computationally expensive, in the s-domain the theorem can be used to formulate a damage location algorithm that is simple and computationally efficient. The key idea is that the Laplace transform of the stress field, which is also zero over the damaged region, can be computed directly from $\ell_{(s)}$. In the Laplace domain formulation damage localization information is obtained from each value of s where the transfer matrices are evaluated.

To formalize assume that d independent stress components are needed to describe the stress field in finite element j, that the matrix relating these stresses to the element degrees of freedom is E_j and that the degrees of freedom in the element are related to those in the structure by N_j. The time history of the stresses on this element for an arbitrary loading can then be written as

$$\sigma_{j,(t)} = E_j N_j y_{(t)} \tag{42}$$

with $\sigma_j \in R^{d\times 1}$, $E_j \in R^{d\times \text{ne}}$, and $N_j \times R^{\text{ne}\times n}$. By definition, ne is the number of DOF in the element stiffness matrix and $d = \text{ne} - \text{rbm}$, where rbm is the number of rigid body modes. From Eq. (42) one can write

$$\sigma_{j,(t)} = E_j N_j \int_0^t H(t-\tau) f(\tau)\, \mathrm{d}\tau \tag{43}$$

where H is the impulse response matrix and f is the loading, which at this point is arbitrary. Taking a Laplace transform of Eq. (43) and recognizing that the Laplace transform of H is the transfer matrix G_m one gets

$$\sigma_{j,(s)} = E_j N_j G_m f \tag{44}$$

Consider now the case where f is a static load; for this situation one has

$$\sigma_j = E_j N_j K^{-1} f \tag{45}$$

Comparing Eqs. (44) and (45) one concludes that the Laplace transform of the stress field can be obtained by performing a static analysis on the system using a load given by

$$L_{\text{st}} = K G_m f \tag{46}$$

Note the transfer matrix in Eq. (46) is that for the model of the undamaged state and it is understood that for the damage localization $f = \ell_{(s)}$, with zeros added at the coordinates that are not part of the experimental set. The fact that the Laplace transform of the stress field is zero over the damaged region follows from the fact that the time history is zero when $f = \ell$.

A. Implementation Issues

When applying the DDLV theorem to real data it is necessary to operate using modally truncated transfer matrices and with models of the undamaged state that inevitably differ from the actual structure. In this regard, a significant result is the fact that the differences between ideal conditions and those that prevail in practice affect the DDLV computations in a way that has a systematic s-dependence. The rest of this section offers specific recommendations on where to evaluate ΔG and on how to amalgamate results obtained from different s-locations.

1. s-Selection

Operating close to the identified poles is beneficial for controlling truncation error, but s-values that are too close to the poles are not good selections because, for these, the localizing stress field can be unduly distorted by discrepancies between the model and the real structure. A reasonable approach is to select s-values that are as close to the identified poles (of the undamaged state) as some proximity constraint permits. In this regard, analysis based on assumptions on the nature and level of the model error show that a reasonable criterion is to exclude s-values that belong to the interior of circles centered at the identified poles and having radii of $0.12s_i$, where s_i is the imaginary part of each pole. Moreover, to ensure that s is not closer to the first truncated pole than to the last identified one it is appropriate to limit $\mathrm{Imag}(s)$ to no more than $\mathrm{Imag}(s_{w-1})$ where s_w is the last identified pole (assuming one is working on the upper half plane, that the lower modes are the ones identified and that the poles are ordered using their imaginary part).

2. Effective Null

Another issue is whether ΔG can be treated as rank deficient. This question was examined for the particular case $s = 0$ in [12] and a procedure to decide which singular vectors can be treated as belonging to the effective null space was developed there. Specifically, an index designated as svn is computed for each singular vector and those vectors for which $svn \leq 0.2$ are assumed to have damage localization information. Noting that the rank of the exact ΔG is not s-dependent [see Eq. (25)] we make the operational assumption that if the vector associated with the smallest singular value of ΔG satisfies the svn criterion at $s = 0$ then the last vector has damage locating properties at any s (see preceding subsection). Although other singular vectors of ΔG may have damage locating properties we have opted to use only the last one on the premise that it is the most accurate DDLV and that multiple vectors are best obtained by changing s. The steps to compute the svn index are summarized in the Appendix.

3. Discriminating Between Potentially Damaged and Undamaged Elements

Given that errors in ΔG and in the model lead to nonzero stresses in the damaged elements, a procedure for deciding when the field is sufficiently small to place an element in the potentially damaged set is needed. We outline two alternatives: one based on a weighted stress index (WSI) [12] and the other on a subspace angle index (SAI). Both indices and the associated discriminating criteria are simply stated in equation form and we do so without preamble, namely, with PDS as the "potentially damaged set" one has

WSI-Criterion

$$\text{PDS} = \{\text{elements} \mid \text{WSI} \leq \text{tol}\} \tag{47}$$

$$\text{WSI} = \frac{\sum_{j=1}^{\text{ns}} \text{nsi}}{svn \cdot \text{ns}} \tag{48}$$

$$\text{tol} = \max\left(1, \frac{\text{WSI}_{\max}}{10}\right) \tag{49}$$

where ns = number of s values used, svn = index from the Appendix, nsi = characterizing stress in the element normalized so that, for each s value, the largest over the structure is unity. The characterizing stress is positive and should be defined such that it is zero only if the strain energy in the element is zero.

SAI-Criterion

$$\text{PDS} = \{\text{elements}|\text{SAI} \leq 0.1\} \tag{50}$$

$$\text{SAI} = \frac{\sum_{j=1}^{\text{ns}} \theta_j}{\sum_{j=1}^{\text{ns}} \theta_j|_{\max}} \tag{51}$$

where θ = subspace angle between the null space of the stiffness matrix for the finite element considered and the element nodal displacements induced by the DDLV. We note that the bulk of the numerical experience at present is connected with the use of WSI [12].

B. Summary of the s-Domain DDLV Approach

1) Select a value of s in the complex plane (see discussion in preceding section on recommended s-selection). In the first step take $s = 0$.

2) Derive the transfer matrices G and $\tilde{G}$; subtract to obtain ΔG.

3) If $s = 0$ compute svn from the Appendix (if $svn > 0.2$ the nulls space localization does not apply; in this case damage is either too spatially extensive for the number of sensors or the accuracy of ΔG is not adequate). Assuming $svn \leq 0.2$ take ℓ as the vector associated with the smallest singular value of ΔG.

4) Expand ℓ with zeros to the full set of coordinates for the model and treat it as f in Eq. (46) to compute L_{st}.

5) Compute the s-domain response by applying L_{st} to the structural model as a static load (in general the stress field and the nodal displacements are complex).

6) Repeat from step one for as many s-values as desired.

7) Select the PDS using the criteria in Eq. (47) or in Eq. (50).

C. Example 1

This example illustrates the mechanics of the s-domain implementation in the idealized situation where the information is precise. To allow independent reproduction with ease we select the simple structure (and damage scenario) shown in Fig. 1. Because selection of s is immaterial in the ideal case we choose, arbitrarily, $s = 1 + 10i$. The transfer matrices defined at the sensors for this s can be easily shown to be

$$G = \begin{bmatrix} 0.241 - 0.633i & -0.0259 + 0.228i \\ \text{sym} & -0.273 - 0.250i \end{bmatrix} \cdot 10^{-2} \tag{52a}$$

$$\tilde{G} = \begin{bmatrix} 0.301 - 0.867i & -0.006 + 0.185i \\ \text{sym} & -0.267 - 0.257i \end{bmatrix} \cdot 10^{-2} \tag{52b}$$

and subtracting one gets

$$\Delta G = \begin{bmatrix} 6.035 - 23.413i & 2.046 - 4.280i \\ \text{sym} & 0.555 - 0.750i \end{bmatrix} \cdot 10^{-4} \tag{53}$$

As one anticipates from the theory (because $q = 2$ and $p = 1$) the matrix ΔG is rank one and computations show that a vector in the null space is

$$\ell = \begin{Bmatrix} 0.1925 \\ -0.963 + 0.189i \end{Bmatrix} \tag{54}$$

Expanding the vector in Eq. (54) to the full set of coordinates by adding a zero at coordinate 2 and substituting the result into Eq. (46) one gets

$$L_{\text{st}} = \begin{Bmatrix} 0.0283 - 0.346i \\ -0.277 - 0.578i \\ 0.277 + 0.578i \end{Bmatrix} \tag{55}$$

for which a standard static analysis gives

$$\text{Forces in Springs} = \begin{Bmatrix} 0.0283 - 0.346i \\ 0 \\ 0.277 + 0.579i \end{Bmatrix} \tag{56}$$

illustrating that damage is on the second spring (computation of WSI in this ideal situation is superfluous so we do not include it).

D. Example 2

This example considers a plate represented by the finite element model shown in Fig. 4. The plate, which has free-free boundaries, was used in [18] to explore how the distance from the measurement points to the damage affected the capability of an algorithm to locate damage based on updating using FRFs. The most difficult case considered was that in which sensors are available along the 11 nodes in line AB and the damage takes place in element 29 (90% reduction in stiffness). For this situation the algorithm tried in [18] proved unsuccessful, even though noise and modeling error were not included in the analysis.

The results of applying the DDLV theorem are presented in Fig. 5 in terms of SAI computed using results from the four (arbitrarily selected) values of s listed on the figure title. As can be seen, the damaged is sharply located by SAI = 0 in element 29. This exact result can be theoretically anticipated because in the model of Fig. 4 each finite element has nine deformation modes (12 DOF and three rigid body modes) so, for one damaged element, ΔK is rank 9, and because there are 11 sensors ΔG is guaranteed to be rank deficient.

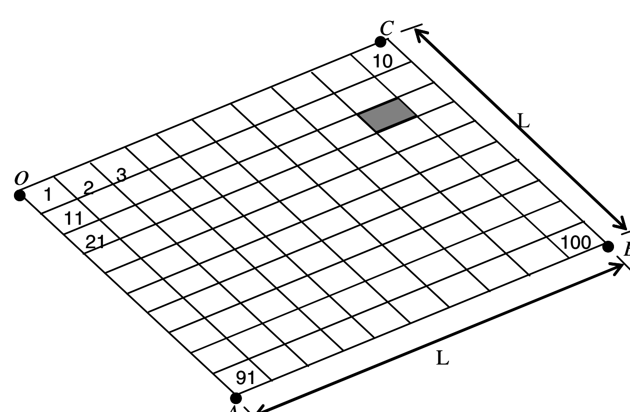

Fig. 4 Plate with measurement points and damaged element used in example 2.

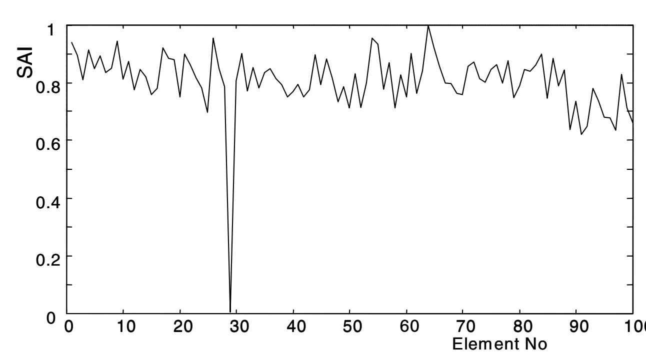

Fig. 5 SAI computed using (s = 60i, 90i, 125i, and 150i).

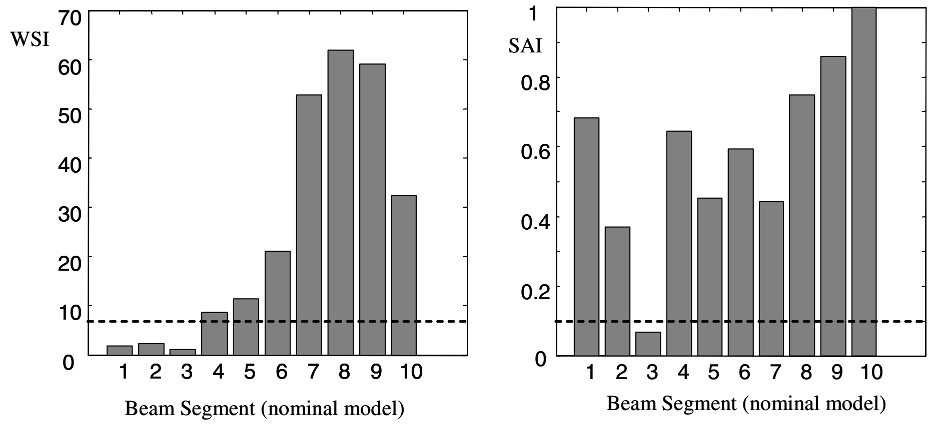

Fig. 6 WSI and SAI for example 3 (cutoff to select PDS shown dotted).

E. Example 3

In this final example modal truncation and model error are considered. The structure is a simply supported prismatic beam with four equally spaced sensors measuring translational motion normal to the beam axis. The truth model (which replaces the "real structure" in the analysis) is taken as a FE representation having 50 equal-sized elements, a consistent mass matrix (100-DOF) and 2% modal damping. The first frequency in the undamaged state is 1 Hz. The nominal model is a lumped mass 9-DOF model that considers translational inertia only. Truncation is incorporated by assuming that only the first five modes (10 complex pairs) are identified in the reference and the damaged states. Damage is simulated as a reduction of EI by 40% in elements 11–15 in the truth model, which corresponds to the third element in the nominal model. A simple approach for selecting s-values in lightly damped structures without having to look explicitly at the exclusion circles is to treat the problem as if the poles were on the imaginary line. With this assumption one concludes (from the discussion in the s-selection section) that s-values located on the line segment $s = 0.12\beta + \beta i$, where β is positive and no larger than the first to last identified pole are adequate. In this example $0 \leq \beta \leq 100.5$ rad/s. Results for WSI and SAI computed using 10 equally spaced s-values on the noted segment are shown in Fig. 6.

As can be seen, both discriminating criteria correctly identifying the third segment of the nominal model as being in the PDS, although in this example SAI offers much better sharpness and resolution than WSI.

F. Scalability

An important issue with any method for damage characterization is whether the operating assumptions restrict the scale at which the method can operate. In the DDLV approach there are no scalability concerns arising from increasing complexity of the model or discrepancies between the number of sensors and the number of DOFs but appear in relation to the possibility of extracting DDLVs from the data. Indeed, in practice ΔG is approximate due to truncation and errors in the identified modes obtained in the reference and the damaged state and the matrix may prove, for some conditions, inadequate for computing DDLVs. In this regard it is opportune to note that when accuracy in ΔG is inadequate the issue tends to be self-diagnosed by a high *svn* index and by the fact that results obtained at different s-values do not point to the same damaged elements so the thresholds on WSI and SAI are not anywhere satisfied and the PDS proves to be an empty set. Worth noting explicitly is that because the DDLV works with ΔG and this matrix is a difference, accuracy does not require that a high number of modes be available but rather that the damage have little effect on the truncated modes.

VI. Conclusions

The DDLV theorem provides a means to map changes in transfer matrices to the location of stiffness related damage. The theorem applies equally to single or multiple damage scenarios and is general with regards to the type of structure considered (i.e., beam, truss, plate, etc.). Worth restating is the fact that the DDLV method (built from the theorem) extends the null space strategy introduced in [12] to interrogate changes in flexibility to the dynamic domain. Benefits derived from interrogating ΔG include potential gains in resolution and damage observability and the fact that assessments at different s-values allow creation of a rich aggregate of results that can be used to filter stochastic error and improve robustness. As there is little that comes without a price tag, however, the additional information in ΔG also comes with a larger domain of uncertainties because (in contrast with the situation when changes in flexibility are considered) inertia and damping have to be modeled to compute the localizing stress field. The paper discusses how undue distortions from truncation and model errors can be avoided by appropriate selection of where in the s-plane ΔG is evaluated.

The DDLV theorem shows that localization of stiffness related damage can be decoupled from damage severity without regard to any differences between the number of DOF in the model (which in a theoretical discussion replaces the structure) and the number of measured coordinates. Statements often made in the literature indicating that the mismatch between the two sets has to be resolved to obtain inverse solutions that point to the location of damage are, therefore, not factual.

Appendix: Computation of the *svn* Index

Performing a SVD of the change in the transfer matrix at the origin one has

$$\Delta G = VSV^T \tag{A1}$$

with

$$\begin{aligned} V &= \lfloor v_1 \quad \cdots \quad v_j \quad \cdots \quad v_q \rfloor \quad \text{and} \\ S &= \mathrm{diag}\lfloor s_1 \quad \cdots \quad s_j \quad \cdots \quad s_q \rfloor \end{aligned} \tag{A2}$$

The DDLV approach applies if

$$\mathrm{svn} = \sqrt{\frac{c_q^2 s_q}{(c_j^2 s_j)_{\max}}} \leq 0.2 \tag{A3}$$

where c_j is the constant needed to make the largest magnitude of the characterizing stress over any element for the v_j loading equal to one. In the s-domain implementation of the DDLV theorem the stress field is complex but the characterizing stress is taken as a magnitude. The interested reader can review the rational used to arrive at the criterion in Eq. (A3) in Bernal [12].

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